Chaos in the thermodynamic Bethe ansatz

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Abstract: We investigate the discretized version of the thermodynamic Bethe ansatz equation for a variety of 1+1 dimensional quantum field theories. By computing Lyapunov exponents we establish that many systems of this type exhibit chaotic behaviour, in the sense that their orbits through fixed points are extremely sensitive with regard to the initial conditions.

1. Introduction

The thermodynamic Bethe ansatz (TBA) equation [1, 2] is an important tool in the context of 1+1 dimensional integrable quantum field theories. It serves to extract various types of informations, such as the Virasoro central charge of the underlying ultraviolet conformal field theory [3], vacuum expectation values [1, 4] etc. As it is a nonlinear integral equation, it can be solved analytically only in very few circumstances. In general, one relies on numerical solutions of its discretised version

$$\varepsilon_A^{n+1}(\theta) = r m_A \cosh\theta - \sum_B \int_{-\infty}^{\infty} d\theta' \varphi_{AB}(\theta-\theta') \ln\left(1+e^{-\varepsilon_B^n(\theta')}\right) \quad . \tag{1.1}$$

Here r is the inverse temperature, m_A is the mass of a particle of type A, θ is the rapidity and φ_{AB} denotes the logarithmic derivative of the scattering matrix between the particles of type A and B. The unknown quantities in these equations are the pseudo-energies $\varepsilon_A(\theta)$. The standard solution procedure for (1.1) consists of a consecutive iteration of the equation with initial values $\varepsilon_A^0(\theta) = r m_A \cosh\theta$. At the heart of this procedure lie the *assumptions* that the exact solution is reached for $n \to \infty$, i.e. the sequence converges, and furthermore that the final answer is non-sensitive with regard to the initial values, that is its uniqueness. In general these assumptions are poorly justified and only few rigorous investigations for some simple models exist [5, 2, 6]. So far the outcome has always been

that these assumptions on the existence and uniqueness of the solution indeed hold, albeit for certain systems convergence problems in certain regimes have been noted [7, 8]. The main purpose of this note is to show that this believe has to be challenged and is in fact unjustified for certain well defined theories. We note that our findings do neither effect the principles of the TBA itself nor the consistency of the quantum field theory it is meant to investigate. However, they indicate that one needs to be very cautious when using the above solution procedure and making deductions about the physics for such theories as one might just be mislead by the non-convergence of the mathematical procedure used to solve the TBA-equations.

Here we will not analyze the full TBA-equations (1.1), but rather concentrate on the ultraviolet regime, that is $r \approx 0$, in which it possess some approximation. Clearly, the occurrence of chaotic behaviour in this regime will have consequences for the finite temperature regime. We encounter the interesting phenomenon that the iterative procedure is convergent beyond the ultraviolet (or for a certain choice of parameters in some theories), but that unstable fixed points are present in the ultraviolet, meaning that this regime can never be reached by the iterative solution procedure for (1.1).

2. Stability of fixed points and Lyapunov exponents

For completeness, let us first briefly recall a few well known basic facts concerning the nature of fixed points which may be found in standard text books on dynamical systems, see e.g. [9]. The objects of our investigations are difference equations of the type

$$\vec{x}_{n+1} = \vec{F}(\vec{x}_n), \tag{2.1}$$

where $n \in \mathbb{N}_0$, $\vec{x}_n \in \mathbb{R}^\ell$ and $\vec{F} : \mathbb{R}^\ell \to \mathbb{R}^\ell$ is a vector function. We are especially interested in the fixed points $\vec{x}_f$ of this system being defined as

$$\vec{F}(\vec{x}_f) = \vec{x}_f \ . \tag{2.2}$$

The fixed point is reached by iterating (2.1), if for a perturbation of it, defined as $\vec{y}_n = \vec{x}_n - \vec{x}_f$, we have $\lim_{n\to\infty} \vec{y}_n = 0$. From (2.1) we find

$$\vec{y}_{n+1} + \vec{x}_f = \vec{F}(\vec{y}_n + \vec{x}_f) = \vec{F}(\vec{x}_f) + J \cdot \vec{y}_n + \mathcal{O}(|\vec{y}_n|^2) \quad \text{for } \vec{y}_n \to 0 \tag{2.3}$$

where J is the $\ell \times \ell$ Jacobian matrix of the vector function $\vec{F}(\vec{x})$

$$J_{ij} = \left. \frac{\partial F_i}{\partial x_j} \right|_{\vec{x}_f} \qquad\qquad \text{for } 1 \leq i, j \leq \ell. \tag{2.4}$$

The linearized system which arises from (2.3) for $\vec{y}_n \to 0$

$$\vec{y}_{n+1} = J \cdot \vec{y}_n \tag{2.5}$$

governs the nature of the fixed point under certain conditions [9]. Evidently, it is solved by

$$\vec{y}_n = q_i^n \vec{v}_i \qquad \text{with} \qquad J \cdot \vec{v}_i = q_i^n \vec{v}_i \quad \text{for } 1 \leq i \leq \ell \ . \tag{2.6}$$

Excluding the case when the eigenvectors $\vec{v}_i$ of the Jacobian matrix are not linearly independent, we can expand the initial value uniquely

$$\vec{y}_0 = \sum_{i=1}^{\ell} \varsigma_i \vec{v}_i \tag{2.7}$$

such that

$$\vec{y}_n = \sum_{i=1}^{\ell} \varsigma_i q_i^n \vec{v}_i \ . \tag{2.8}$$

It is now obvious from (2.8) that the perturbation of the fixed point $\vec{y}_n$ will grow for increasing n if $|q_i| > 1$ for some $i \in \{1, \ldots, \ell\}$. In that case the fixed point $\vec{x}_f$ is said to be linearly unstable. On the other hand, if $|q_i| < 1$ for all $i \in \{1, \ldots, \ell\}$ the perturbation will tend to zero for increasing n and the fixed point $\vec{x}_f$ is said to be linearly stable. It can be shown that under some conditions [9] the fixed points are nonlinearly stable when they are linearly stable.

In general, that is for any point $\vec{x}$ rather than just the fixed points $\vec{x}_f$, stability properties are easily encoded in the Lyapunov exponents λ_i. Roughly speaking the Lyapunov exponents are a measure for the exponential separation of neighbouring orbits. One speaks of unstable (chaotic) orbits if $\lambda_i > 0$ for some $i \in \{1, \ldots, \ell\}$ and stable orbits if $\lambda_i < 0$ for all $i \in \{1, \ldots, \ell\}$. For an arbitrary point $\vec{x}$ the ℓ Lyapunov exponents for the above mentioned system (2.1) are defined as

$$\lambda_i = \lim_{n\to\infty} \left[\frac{1}{n} \ln \left| q_i[\vec{F}^n(\vec{x})] \right| \right] = \lim_{n\to\infty} \left[\frac{1}{n} \sum_{k=0}^{n-1} \ln \left| q_i[\vec{F}^k(\vec{x})] \right| \right], \tag{2.9}$$

where the $q_i\,(\vec{x})$ are the eigenvalues of the Jacobian matrix as defined in (2.6), but now at some arbitrary point $\vec{x}$. Taking the point to be a fixed point, we can relate (2.9) to the above statements. At the fixed point we have of course $\vec{F}^k(\vec{x}_f) = \vec{x}_f$, such that

$$\lambda_i = \lim_{n\to\infty} \left[\frac{1}{n} \sum_{k=0}^{n-1} \ln |q_i\,(\vec{x}_f)| \right] = \ln |q_i| \ . \tag{2.10}$$

Therefore, a stable fixed point is characterized by $\lambda_i < 0$ or $|q_i| < 1$ for all $i \in \{1, \ldots, \ell\}$ and an unstable fixed point by $\lambda_i > 0$ or $|q_i| > 1$ for some $i \in \{1, \ldots, \ell\}$. We can now employ this criterion for some concrete systems.

3. Unstable fixed points in constant TBA equations

We adopt here the notation of [10, 11, 12], by which a large class of integrable quantum field theories can be referred to in a general Lie algebraic form as $\mathbf{g}|\tilde{\mathbf{g}}$-theories. Their underlying ultraviolet conformal field theories can be described by the theories investigated in [13, 14, 15] (and special cases thereof) with Virasoro central charge $c = \ell\tilde{\ell}\tilde{h}/(h+\tilde{h})$. Here $\ell(\tilde{\ell})$ and $h(\tilde{h})$ are the rank and the Coxeter number of $\mathbf{g}(\tilde{\mathbf{g}})$, respectively. In particular, $\mathbf{g}|\mathbf{A}_1$ is identical to the minimal affine Toda theories (ATFT) [16, 17] and $\mathbf{A}_n|\tilde{\mathbf{g}}$ corresponds

to the $\tilde{\mathbf{g}}_{n+1}$-homogeneous Sine-Gordon (HSG) models [18, 19]. In this formulation each particle is labelled by two quantum numbers (a, i), which take their values in $1 \leq a \leq \ell$ and $1 \leq i \leq \tilde{\ell}$. Hence, in total we have $\tilde{\ell} \times \ell$ different particle types. It is a standard procedure in this context [1, 2] to approximate the pseudo-energies in (1.1) by $\varepsilon_a^i(\theta) = \varepsilon_a^i = \text{const}$ in a large region for θ when r is small. For convenience one then introduces further the quantity $x_a^i = \exp(-\varepsilon_a^i)$ such that (1.1) can be cast into the compact form

$$x_a^i = \prod_{b=1}^{\ell} \prod_{j=1}^{\tilde{\ell}} (1 + x_b^j)^{N_{ab}^{ij}} =: F_a^i(\vec{x}) \qquad \text{with } N_{ab}^{ij} = \delta_{ab}\delta_{ij} - K_{ab}^{-1}\tilde{K}_{ij}\,. \tag{3.1}$$

The matrix N_{ab}^{ij} in (3.1) encodes the information on the asymptotic behaviour of the scattering matrix. As stated in (3.1) it is specific to each of the $\mathbf{g}|\tilde{\mathbf{g}}$-theories with K and $\tilde{K}$ being the Cartan matrix of $\mathbf{g}$ and $\tilde{\mathbf{g}}$, respectively. The equations (3.1) are referred to as the constant TBA-equations. They govern the ultraviolet behaviour of the system and their solutions yield directly the effective central charge

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{a=1}^{\ell} \sum_{i=1}^{\tilde{\ell}} \mathcal{L}\left(\frac{x_a^i}{1 + x_a^i} \right) \tag{3.2}$$

with $\mathcal{L}(x) = \sum_{n=1}^{\infty} x^n/n^2 + \ln x \ln(1-x)/2$ denoting Rogers dilogarithm (see e.g. [20] for properties).

Let us now discretise (3.1) and analyze it with regard to the nature of its fixed points. According to the argument of section 2, we have to compute first of all the Jacobian matrices of $\vec{F}$

$$J_{ab}^{ij} = \left. \frac{\partial F_a^i}{\partial x_b^j} \right|_{\vec{x}_f} = N_{ab}^{ij} \frac{(x_a^i)_f}{1 + (x_b^j)_f}\,. \tag{3.3}$$

Next we need to determine the eigensystem of the Jacobian matrix. This is not possible to do in a completely generic way at present, since not even the solutions, i.e. fixed points, of (3.1) are known in a general fashion. Instead, we present some examples to exhibit the possible types of behaviour.

3.1 Stable fixed points

We start with a simple example of a stable fixed point. We present the $A_2|A_1$ case, which after the free Fermion $(A_1|A_1)$ is the next non-trivial example in the series of the minimal ATFTs, the scaling three-state Potts model with Virasoro central charge $c = 4/5$. The TBA has been investigated in [1, 2]. The constant TBA-equations (3.1)

$$x_1 = (1 + x_1)^{-1/3}(1 + x_2)^{-2/3} = F_1(\vec{x}), \qquad x_2 = (1 + x_1)^{-2/3}(1 + x_2)^{-1/3} = F_2(\vec{x}), \tag{3.4}$$

can be solved analytically by the golden ratio $\tau := (\sqrt{5} - 1)/2 = x_1 = x_2$. Using this solution for the fixed point $\vec{x}_f = (x_1, x_2)$, we compute the Jacobian matrix of $\vec{F}(\vec{x})$

$$J(\vec{x}_f) = -\frac{1}{3} \begin{pmatrix} \tau^2 & 2\tau^2 \\ 2\tau^2 & \tau^2 \end{pmatrix}, \tag{3.5}$$

with eigensystem

$$q_1 = -\tau^2 \approx -0.38197, \quad \vec{v}_1 = (1,1), \quad q_2 = \tau^2/3 \approx 0.12732, \quad \vec{v}_2 = (-1,1) \ .$$

As the eigenvectors are obviously linearly independent and $|q_1| < 1, |q_2| < 1$, we deduce that all Lyapunov exponents are negative and therefore that the fixed point is stable. Indeed, numerical studies of this system exhibit a fast convergence and a non-sensitive behaviour with regard to the initial values of the iterative procedure.

For some theories the solutions are known analytically in a closed form. For instance, the constant TBA equations for the $A_1|A_\ell$-theories ($\equiv SU(\ell+1)_2$-HSG-model) are solved by

$$x_1^i = \left[\frac{\sin[\pi(1+i)\lambda]}{\sin(\pi\lambda)} \right]^2 - 1, \qquad \text{for } 1 \leq i \leq \ell \tag{3.6}$$

with $\lambda = 1/(3+\ell)$ [21, 22, 23, 12]. Taking this solution for the fixed point we compute the Jacobian matrix (3.3) with $N_{ij} = (\delta_{i,j+1} + \delta_{i,j-1})/2$ to

$$J_{11}^{ij}(\vec{x}_f) = \frac{1}{2} \left[\frac{\sin[\pi(2+i)\lambda]}{\sin(i\pi\lambda)} \delta_{i,j+1} + \frac{\sin(i\pi\lambda)}{\sin[\pi(2+i)\lambda]} \delta_{i,j-1} \right] . \tag{3.7}$$

and the eigenvalues to $q_i = \cos[\pi(i+1)\lambda]$. As $|q_i| < 1$ for all $i \in \{1, \ldots, \ell\}$ the fixed points are stable. We investigated various minimal affine Toda field theories which all posses fixed points of this nature. In fact, the general assumption is that all systems exhibit such a behaviour. We present now some counter examples which refute this believe.

3.2 Stable two-cycles

We start with a system which does not possess a stable fixed point, but rather a stable two-cycle, i.e. a solution for

$$\vec{G}(\vec{x}) := \vec{F}(\vec{F}(\vec{x})) = \vec{x} \ . \tag{3.8}$$

We consider the $A_2|A_2$-theories ($\equiv SU(3)_3$-HSG model) studied already previously by means of the TBA in [24]. Its extreme ultraviolet Virasoro central charge is $c = 2$. The constant TBA-equations for this case read

$$x_1^1 = \frac{(1+x_1^2)^{2/3}(1+x_2^2)^{1/3}}{(1+x_1^1)^{1/3}(1+x_2^1)^{2/3}} = F_1(\vec{x}), \quad x_2^1 = \frac{(1+x_1^2)^{1/3}(1+x_2^2)^{2/3}}{(1+x_1^1)^{2/3}(1+x_2^1)^{1/3}} = F_2(\vec{x}), \tag{3.9}$$

$$x_1^2 = \frac{(1+x_1^1)^{2/3}(1+x_2^1)^{1/3}}{(1+x_1^2)^{1/3}(1+x_2^2)^{2/3}} = F_3(\vec{x}), \quad x_2^2 = \frac{(1+x_1^1)^{1/3}(1+x_2^1)^{2/3}}{(1+x_1^2)^{2/3}(1+x_2^2)^{1/3}} = F_4(\vec{x}), \tag{3.10}$$

with analytic solution $x_1^1 = x_2^1 = x_1^2 = x_2^2 = 1$. Taking this solution as the fixed point, we compute the Jacobian matrix for $\vec{F}(\vec{x})$

$$J(\vec{x}_f) = \frac{1}{6} \begin{pmatrix} -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \\ 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \end{pmatrix} \tag{3.11}$$

with eigensystem

$$\begin{array}{llll} q_1 = -1, & \vec{v}_1 = (-1,-1,1,1), & q_2 = 1/3, & \vec{v}_2 = (-1,1,-1,1), \\ q_3 = 0, & \vec{v}_3 = (1,0,0,1), & q_4 = 0, & \vec{v}_4 = (0,1,1,0). \end{array} \tag{3.12}$$

We observe that there is one eigenvalue with $|q_1| = 1$, which is generally called a marginal behaviour, i.e. the stability properties depend on the other eigenvalues and on the next leading order. In fact, we can see from (2.8) that the perturbation of the fixed point will remain the same even for large values of n, flipping between two values and thus suggesting the existence of a stable two cycle (3.8). We find that (3.8) can be solved by

$$x_1^1 = x_2^1 = 1/x_1^2 = 1/x_2^2 = \kappa, \tag{3.13}$$

for any arbitrary value of κ. To determine the stability of the two-cycle we have to compute the Jacobian matrix for $G(\vec{x})$

$$J(\vec{x}_f) = \frac{1}{9+9\kappa}\begin{pmatrix} 5\kappa & 4\kappa & -4\kappa^2 & -5\kappa^2 \\ 4\kappa & 5\kappa & -5\kappa^2 & -4\kappa^2 \\ -4/\kappa & -5/\kappa & 5 & 4 \\ -5/\kappa & -4/\kappa & 4 & 5 \end{pmatrix} \tag{3.14}$$

which has eigensystem

$$\begin{array}{llll} q_1 = 1, & \vec{v}_1 = (-\kappa^2,-\kappa^2,1,1) & q_2 = 1/9, & \vec{v}_2 = (-\kappa^2,\kappa^2,-1,1), \\ q_3 = 0, & \vec{v}_3 = (\kappa,0,0,1), & q_4 = 0, & \vec{v}_4 = (0,\kappa,1,0). \end{array} \tag{3.15}$$

We conclude from this that one approaches a stable two-cycle when iterating the discretised version of (3.8). Thus, we note that the TBA-system for the $SU(3)_3$-HSG model in the ultraviolet regime does not posses a stable fixed point but an infinite number of stable two-cycles of the type (3.13). It is now intriguing to note that when using this solution to compute the effective Virasoro central charge (3.2), one always obtains the expected value $c = 2$ for any value of $\kappa \in \mathbb{R}$, simply due to an identity for the Rogers dilogarithm $\mathcal{L}(1-x) + \mathcal{L}(x) = \pi^2/6$. Hence, despite the fact, that one is using entirely wrong pseudo-energies, one obtains by pure luck an apparent confirmation of the theories consistency.

3.3 Unstable fixed points, chaotic behaviour

In this section we present some TBA-systems for well-defined quantum field theories, which exhibit a chaotic behaviour in the sense that their iterative solutions are extremely sensitive with regard to the initial values.

3.3.1 $A_4|A_4$

This model is the $SU(5)_5$-HSG model with extreme ultraviolet Virasoro central charge $c = 8$. To reduce the complexity of the model, we exploit already from the very beginning the $\mathbb{Z}_2$-symmetries in the A_4-Dynkin diagrams and identify $x_1^1 = x_1^4 = x_4^1 = x_4^4$, $x_2^2 = x_2^3 =$

$x_3^2 = x_3^3$, $x_2^1 = x_3^1 = x_2^4 = x_3^4$, $x_1^2 = x_4^2 = x_1^3 = x_4^3$. With these identifications the constant TBA-equations can be brought into the form

$$x_1^1 = \frac{(1+x_1^2)(1+x_2^2)}{(1+x_1^1)(1+x_2^1)^2} = F_1(\vec{x}), \qquad x_2^1 = \frac{(1+x_1^2)(1+x_2^2)^2}{(1+x_1^1)^2(1+x_2^1)^3} = F_2(\vec{x}), \tag{3.16}$$

$$x_1^2 = \frac{(1+x_1^1)(1+x_2^1)}{(1+x_2^2)} = F_3(\vec{x}), \qquad x_2^2 = \frac{(1+x_1^1)^2(1+x_2^1)^2}{(1+x_1^2)(1+x_2^2)} = F_4(\vec{x}). \tag{3.17}$$

We can solve these equations analytically by one, τ and $\tilde{\tau} := 1/\tau = (\sqrt{5}+1)/2$

$$\begin{aligned} x_1^1 &= x_1^4 = x_4^1 = x_4^4 = x_2^2 = x_2^3 = x_3^2 = x_3^3 = 1, \\ x_1^2 &= x_4^2 = x_1^3 = x_4^3 = \tilde{\tau}, \\ x_2^1 &= x_3^1 = x_2^4 = x_3^4 = \tau. \end{aligned} \tag{3.18}$$

With this solution for $\vec{x}_f$ at hand we compute the Jacobian matrix of $\vec{F}(\vec{x})$ in (3.16), (3.17)

$$J(\vec{x}_f) = \begin{pmatrix} -1/2 & -2\tau & \tau^2 & 1/2 \\ -\tau & -3\tau^2 & 2\tau - 1 & \tau \\ \tilde{\tau}/2 & 1 & 0 & -\tilde{\tau}/2 \\ 1/2 & 2\tau & -\tau^2 & -1/2 \end{pmatrix}. \tag{3.19}$$

Now we find the eigensystem

$$\begin{aligned} q_1 &= -\tilde{\tau}^2 \approx -2.6180, & \vec{v}_1 &= (-1,-1,1,1), \\ q_2 &= 4\tau - 2 \approx 0.47214, & \vec{v}_2 &= (-1,\tau^2,-\tilde{\tau}^2,1), \\ q_3 &= 0, & \vec{v}_3 &= (1,0,0,1), \\ q_4 &= 0, & \vec{v}_4 &= (-2\tau^2,\tau,1,0). \end{aligned} \tag{3.20}$$

As $|q_1| > 1$ and the eigenvectors are linearly independent, we deduce that the Lyapunov exponent λ_1 is positive and therefore that the fixed point (3.18) is unstable. Indeed, numerical studies of this system exhibit that any small perturbation away from the solution (3.18) will lead to a divergent iterative procedure.

Nonetheless, by some manipulations of (3.16), (3.17) one can find equivalent sets of equations which posses stable fixed points and can be solved by means of an iterative procedure. For example, when simply substituting x_1^1 in $F_2(\vec{x})$ we obtain the equations

$$x_1^1 = F_1(\vec{x}) = F_1'(\vec{x}), \qquad x_2^1 = \frac{x_1^1(1+x_2^2)}{(1+x_1^1)(1+x_2^1)} = F_2'(\vec{x}), \tag{3.21}$$

$$x_1^2 = F_3(\vec{x}) = F_3'(\vec{x}), \qquad x_2^2 = F_4(\vec{x}) = F_4'(\vec{x}), \tag{3.22}$$

which are of this kind. Now all Lyapunov exponents resulting from the Jacobian matrix for $\vec{F}'(\vec{x})$ at the fixed point (3.18) are negative. One should note, however, that even though $\vec{F}'(\vec{x})$ it is easily constructed by trial and error from $\vec{F}(\vec{x})$ for the constant TBA-equations, the equivalent manipulations on the full TBA-equations (1.1) are quite unnatural, albeit not impossible to perform once (3.1) is analyzed.

One of the distinguishing features of the HSG-models is that they contain unstable particles in their spectrum, whose masses are characterized by some resonance parameters

σ_{ij} with $1 \leq i,j \leq \tilde{\ell}$. We can now interpret these parameters as bifurcation parameters as common in the study of chaotic systems and investigate the nature of the fixed points when these parameters are varied. In [7] a precise decoupling rule was provided, which describes the behaviour of the theories when some of the σ's become large and tend to infinity. For $SU(5)_5$ we have for instance the following possibilities

$$\lim_{\sigma_{12}\to\infty} SU(5)_5 = SU(2)_5 \otimes SU(4)_5 \quad \text{or} \quad \lim_{\sigma_{23}\to\infty} SU(5)_5 = SU(3)_5 \otimes SU(3)_5 \ . \tag{3.23}$$

For the algebras involved we found that the fixed point of $SU(4)_5$ is unstable, whereas the fixed points of $SU(3)_5$ and $SU(2)_5$ are stable. For our $SU(5)_5$ example this implies that the fixed point in $SU(3)_5 \otimes SU(3)_5$ will be stable, whereas the fixed point in $SU(2)_5 \otimes SU(4)_5$ will be unstable. In general, we find that, while approaching the ultraviolet from the infrared, once the nature of the fixed point has changed from stable to unstable it remains that way. This behaviour can be encoded naturally in standard bifurcation diagrams, which we present elsewhere.

We also found unstable fixed points for other HSG-models related to simply laced algebras and $\mathbf{g}|\tilde{\mathbf{g}}$-theories which are neither HSG nor minimal ATFT. A priori the behaviour is difficult to predict, e.g. whereas $D_4|D_4$ (see [10] for the solution of (3.1)) and $D_4|A_4$ have unstable fixed points, the fixed point in $D_4|A_2$ is stable.

3.3.2 $A_1|C_2$

This model is the simplest example of an HSG model related to a non-simply laced algebra, namely the $Sp(4)_2$-HSG model with central charge $c = 2$. In the TBA analysis carried out in [7] convergence problems in the ultraviolet regime were already commented upon. In fact, we find here that all HSG models which are related to non-simply laced Lie algebras posses unstable fixed points. The constant TBA-equations for $A_1|C_2$ read

$$x_1^1 = \sqrt{(1+x_1^2)(1+x_3^2)}(1+x_2^2) = F_1(\vec{x}),\ x_1^2 = \frac{1}{(1+x_2^2)}\sqrt{\frac{(1+x_1^1)}{(1+x_1^2)(1+x_3^2)}} = F_2(\vec{x}), \tag{3.24}$$

$$x_2^2 = \frac{(1+x_1^1)}{(1+x_1^2)(1+x_2^2)(1+x_3^2)} = F_3(\vec{x}), \quad x_3^2 = \frac{1}{(1+x_2^2)}\sqrt{\frac{(1+x_1^1)}{(1+x_1^2)(1+x_3^2)}} = F_4(\vec{x}), \tag{3.25}$$

with solutions

$$x_1^1 = 3, \quad x_1^2 = x_3^2 = 2/3 \quad \text{and} \quad x_1^2 = 4/5. \tag{3.26}$$

The corresponding Jacobian matrix for $\vec{F}(\vec{x})$ reads

$$J(\vec{x}_f) = \begin{pmatrix} 0 & 9/10 & 5/3 & 9/10 \\ 1/12 & -1/5 & -10/27 & -1/5 \\ 1/5 & -12/25 & -4/9 & -12/25 \\ 1/12 & -1/5 & -10/27 & -1/5 \end{pmatrix} \tag{3.27}$$

with eigensystem

$$\begin{array}{ll} q_1 \approx -1.3647, & \vec{v}_1 = (-3.53, 1, 1.8104, 1), \\ q_2 \approx 0.3973, & \vec{v}_2 = (-80.2656, 1, -20.2124, 1), \\ q_3 \approx 0.1229, & \vec{v}_3 = (2.1956, 1, -0.9180, 1), \\ q_4 = 0, & \vec{v}_4 = (0, -1, 0, 1). \end{array} . \tag{3.28}$$

Since $|q_1| > 0$ we find a positive Lyapunov exponent and therefore the fixed point $\vec{x}_f$ in (3.26) is unstable.
We also checked explicitly $A_1|C_3, A_1|C_4, A_1|G_2, A_2|B_2$ and found a similar behaviour. Based on these examples we conjecture that the constant TBA-equations (3.1) related to **g**-HSG-models with **g** non-simply laced have unstable fixed points.

4. Conclusions

We showed that the discretised TBA-equations for many well-defined quantum field theories exhibit chaotic behaviour in the sense that their orbits are extremely sensitive with regard to the initial conditions. In particular, we found several examples for HSG-models and $\mathbf{g}|\tilde{\mathbf{g}}$-theories which are neither HSG nor minimal ATFT. Apart from the statements, that all $A_1|A_\ell$-theories have stable fixed points and apparently all HSG-models which are related to non-simply laced models have unstable fixed points, we did not find yet a general pattern which characterizes such theories in a more concise way.

Our findings clearly explain the convergence problems reported upon earlier in [7] and we stress here that they do neither effect the consistency of the quantum field theories nor the validity of the principles underlying the TBA, but only point out the need to solve these theories by alternative means. The closest would be to alter the iterative procedure for (1.1) as indicated in section 3.3.1 for the constant TBA-equations, by defining equivalent sets of equations which have stable fixed points. Unfortunately, we can not settle with these arguments the convergence problems for the models studied in [8], as for those the fixed points are situated at infinity.

Our results clearly indicate that one can only be confident about results obtained from iterating (1.1) if the nature of the fixed points is clarified.

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